Hermite-Hadamard Type Integral Inequalities for Multidimensional General Convex Stochastic Processes

Nurgül Okur Bekar [1], Vildan Karahan [2]

[1] Giresun University, Department of Statistics, Giresun/Turkey and

nrgokur@gmail.com

[2] Institute of Sciences, Giresun University, 28200 Giresun, Turkey.

vildan.karahan2@ugurokullari.k12.tr

ABSTACT. The main subject of this study is initially to consider general (φ-convex) stochastic processes on n-dimensional interval. Besides, some generalized Hermite-Hadamard type inequalities are obtained for these processes.

Keywords: φ-convexity, stochastic processes, n-coordinates, mean-square integral, Hermite-Hadamard inequality.

AMS Classification 2010: 37A50, 32F17.

1. INTRODUCTION

It is well known that, for every real convex function f on the interval $[a, b]$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Alomari [1] generalized the classical Hermite-Hadamard type inequalities for every convex function and for every positive convex function on $[a, b]$. Mwaeze [2] obtained some generalized Hermite-Hadamard's inequality for every convex functions and for every positive convex function on the coordinates. Moreover, the Hermite-Hadamard type inequalities for φ-convex functions were obtained by many researchers as seen in [11]-[14]. For example, Cristescu [11] have presented a version Hermite-Hadamard type inequality for φ-convex functions as follows:

$$f\left(\frac{\varphi(u)+\varphi(v)}{2}\right) \leq \frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)} f(x)dx \leq \frac{f\big(\varphi(u)\big)+f\big(\varphi(v)\big)}{2}.$$

Concordantly, there are satisfactory evidence on similar results belong to stochastic processes. Then, let us see some studies in the literature [3]-[10]. Firstly, convex stochastic processes were proposed and some properties were given for classical convex stochastic processes by Nikodem in [3]. Stochastic convexity and its applications were defined by Shaked et al in [4]. Jensen-convex, λ-convex stochastic processes were introduced by Skowronski in [5]. The classical Hermite-Hadamard inequality to convex stochastic processes was extended by Kotrys in [6]. In recent years, Set et al in [7] obtained some inequalities for coordinated convex stochastic processes. Also, the Hermite-Hadamard type inequalities for harmonically convex stochastic processes on the coordinates were obtained by Okur et al in [8]. Sarikaya et al [9] defined the φ_h-convex stochastic processes and presented Hermite-Hadamard type inequality for these processes. Karahan et al [14] investigated the convex stochastic processes on n-coordinates, and obtained Hermite-Hadamard type inequality for these processes.

The authors' findings led to our motivation to build our work. The main subject of this paper is to obtain some generalized Hermite-Hadamard type inequalities for φ-convex stochastic processes on the n-coordinates.

2. PRELIMINARIES

There are well-known definitions of stochastic processes and some fundamentals about Hermite Hadamard inequality for stochastic processes in the literature (see, [6], [9], [14]).

Definition 2.1[6]. *Let $(\Omega, \Im, P)$ be an arbitrary probability space and $I \in \Im$, $I \subset \mathbb{R}$ be an interval. The stochastic process $X: I \times \Omega \to \mathbb{R}$ is called almost everywhere convex if*

$$X(\lambda t + (1-\lambda)s, \cdot) \leq \lambda X(t, \cdot) + (1-\lambda)X(s, \cdot)$$

for all $t, s \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed then X is said to be concave.

Definition 2.2[6]. *Let $(\Omega, \Im, P)$ be an arbitrary probability space and $I \in \Im$, $I \subset \mathbb{R}$ be an interval. We say that a stochastic process $X: I \times \Omega \to \mathbb{R}$ is called*

(i) *continuous in probability on I if for all $t_0 \in I$ if*

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P-\lim$ denotes limit in probability,

(ii) *mean-square continuous on I if for all $t_0 \in I$ if*

$$\lim_{t \to t_0} E[X(t, \cdot) - X(t_0, \cdot)]^2 = 0,$$

where $E[\xi(t, \cdot)]$ denotes expectation value of the random variable $X(t, \cdot)$,

(iii) *mean-square differentiable at a point if $t \in I$ if there is a random variable $X'(t, \cdot): I \times \Omega \to \mathbb{R}$ such that*

$$X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

We say that a stochastic process $X: I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I.

Definition 2.3[6]. *Let $(\Omega, \Im, P)$ be an arbitrary probability space and $I \in \Im$, $I \subset \mathbb{R}$ be an interval and $X: I \times \Omega \to \mathbb{R}$ be a stochastic process with $E[X(t,\cdot)^2] \leq \infty$ for all $t \in I$. Let $[0,t] \subset I$, $0 = t_0 < t_1 \ldots t_n = t$ be a partition of $[0,t]$ and $\Theta_k \in [t_{k-1}, t_k]$ arbitrary for $k = 1, \ldots, n$. A random variable $\eta: \Omega \times \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t,\cdot)$ on $[0,t]$ if the following identity holds:*

$$\lim_{n\to\infty} E\left(\sum_{k=1}^{n} X(\Theta_k,\cdot).(t_k - t_{k-1}) - \eta(t,\cdot)\right)^2 = 0.$$

Then we can write almost everywhere

$$\int_0^t X(u,\cdot)du = \eta(t,\cdot).$$

The mean-square integral operator is increasing on $[0,t]$ almost everywhere, that is,

$$X(t,\cdot) \leq Y(t,\cdot) \Rightarrow \int_0^t X(t,\cdot)dt \leq \int_0^t Y(t,\cdot)dt.$$

Definition 2.4[9]. Let us consider a function $\varphi: [u,v] \to [u,v]$, $u < v$. *A stochastic process $X: [u,v] \times \Omega \to \mathbb{R}$ is said to be φ-convex on $[u,v]$ if the following inequality holds almost everywhere*

$$X\left(\left(\lambda\varphi(t) + (1-\lambda)\varphi(s)\right),\cdot\right) \leq \lambda X(\varphi(t),\cdot) + (1-\lambda)X(\varphi(s),\cdot)$$

for all $t, s \in [u,v]$ and $\lambda \in [0,1]$. If the above inequality is reversed then X is said to be φ-concave on $[u,v]$.

Theorem 2.1[9]. *Let $X: [u,v] \times \Omega \to \mathbb{R}$ be a φ-convex on $[u,v], u < v$ and mean-square continuous function $\varphi: [u,v] \to [u,v]$, $\varphi(u) < \varphi(v)$. Then one has the inequality almost everywhere*

$$X\left(\frac{\varphi(u)+\varphi(v)}{2},\cdot\right) \leq \frac{1}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} X(t,\cdot)dt \leq \frac{X(\varphi(u),\cdot) + X(\varphi(v),\cdot)}{2}.$$

For $n \geq 2$, let $u_i, v_i, (i = 1,2,\ldots,n)$ be real numbers such that $u_i < v_i$ for $i = 1,2,\ldots,n$, and the n-dimensional interval $\Lambda^n = \prod_{i=1}^{n}[u_i, v_i] \subseteq [0,\infty)^n$.

In the following we give definition of convexity for stochastic processes on n-coordinates:

Definition 2.5[14]. *A stochastic process $X: \Lambda^n \times \Omega \to \mathbb{R}$ is called convex on n-coordinates if the stochastic processes*

$$X_{t_n}^i(t,\cdot) \coloneqq X\big((t_1,\dots,t_{i-1},t,t_{i+1},\dots,t_n),\cdot\big)$$

are convex on $[u_i, v_i]$ *almost everywhere for* $i = 1,2,\dots,n$.

Definition 2.6[14]. *A stochastic process* $X\colon \Lambda^n \times \Omega \to \mathbb{R}$ *is said to be convex on* Δ^n *if the following inequality holds almost everywhere*

$$X\big((\lambda \boldsymbol{t} + (1-\lambda)\boldsymbol{s}),\cdot\big) \le \lambda X(\boldsymbol{t},\cdot) + (1-\lambda)X(\boldsymbol{s},\cdot)$$

for all $\boldsymbol{t} = (t_1, t_2, \dots, t_n), \boldsymbol{s} = (s_1, s_2, \dots, s_n) \in \Lambda^n$ *and* $\lambda \in [0,1]$. *If the above inequality is reversed then X is said to be concave on* Δ^n.

3. MAIN RESULTS

The main goal of this section is to present Hermite-Hadamard type inequalities for general convex stochastic processes on n-coordinates.

For $n \ge 2$, let $u_i, v_i, (i = 1,2,\dots,n)$ be real numbers such that $u_i < v_i$ for $i = 1,2,\dots,n$, $\Delta^n \coloneqq \prod_{i=1}^n [\varphi(u_i), \varphi(v_i)] \subseteq [0,\infty)^n$. Let us consider a function $\varphi\colon \Delta^n \to \Delta^n$.

In the following we give definition of general convexity for stochastic processes on n-coordinates:

Definition 3.1. *A stochastic process* $X\colon \Delta^n \times \Omega \to \mathbb{R}$ *is called general convex on n-coordinates if the stochastic processes*

$$X_{t_n}^i(t,\cdot) \coloneqq X\left(\big(\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(t),\varphi(t_{i+1}),\dots,\varphi(t_n)\big),\cdot\right) \quad (a.e.)$$

are general convex on $[\varphi(u_i), \varphi(v_i)]$ *almost everywhere for* $i = 1,2,\dots,n$.

Definition 3.2. *A stochastic process* $X\colon \Delta^n \times \Omega \to \mathbb{R}$ *is said to be general convex on* Δ^n *if the following inequality holds almost everywhere*

$$X\left(\big(\lambda\varphi(\boldsymbol{t}) + (1-\lambda)\varphi(\boldsymbol{s})\big),\cdot\right) \le \lambda X(\varphi(\boldsymbol{t}),\cdot) + (1-\lambda)X(\varphi(\boldsymbol{s}),\cdot)$$

for all $\boldsymbol{t} = (t_1, t_2, \dots, t_n), \boldsymbol{s} = (s_1, s_2, \dots, s_n) \in \Delta^n$ *and* $\lambda \in [0,1]$. *If the above inequality is reversed then X is said to be general concave on* Δ^n.

Lemma 3.1. *Every general convex stochastic process* $X\colon \Delta^n \times \Omega \to \mathbb{R}$ *is general convex on n-coordinates almost everywhere but converse is not true.*

Proof. Let $X\colon \Delta^n \times \Omega \to \mathbb{R}$ be general convex on Δ^n. Consider $X_{t_n}^i : [\varphi(u_i), \varphi(v_i)] \times \Omega \to \mathbb{R}$ defined by

$$X_{t_n}^i(t,\cdot) \coloneqq X\left(\big(\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(t),\varphi(t_{i+1}),\dots,\varphi(t_n)\big),\cdot\right) \qquad (a.e.)$$

for $t \in [u_i, v_i]$. Now for $t, s \in [u_i, v_i]$ and $\lambda \in [0,1]$ almost everywhere

$$X_{t_n}^i\left(\left(\lambda\varphi(t)+(1-\lambda)\varphi(s)\right),\cdot\right)$$

$$= X\left(\left(\varphi(t_1),\dots,\varphi(t_{i-1}),\lambda\varphi(t)+(1-\lambda)\varphi(s),\varphi(t_{i+1}),\dots,\varphi(t_n)\right),\cdot\right)$$

$$\leq \lambda X\left(\left(\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(t),\varphi(t_{i+1}),\dots,\varphi(t_n)\right),\cdot\right)$$

$$+(1-\lambda)X\left(\left(\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(s),\varphi(t_{i+1}),\dots,\varphi(t_n)\right),\cdot\right)$$

$$= \lambda X_{t_n}^i(\varphi(t),\cdot)+(1-\lambda)X_{t_n}^i(\varphi(s),\cdot)$$

which implies $X_{t_n}^i$ is φ-convex on $[\varphi(u_i),\varphi(v_i)]$, that is, X is general convex on n-coordinates. For converse we give the following counter example:

Example 3.1. Let us consider a stochastic process X: $[0,1]^n \times \Omega \to \mathbb{R}$ defined as

$$X\left(\left(\varphi(t_1),\varphi(t_2),\dots,\varphi(t_n)\right),\cdot\right) = \varphi(t_1)\varphi(t_2)\dots\varphi(t_n).$$

It is a general convex stochastic process on n-coordinates as follows:

$$X_{t_n}^i\left(\left(\lambda\varphi(t)+(1-\lambda)\varphi(s)\right),\cdot\right)$$

$$= \varphi(t_1)\varphi(t_2)\dots\varphi(t_{i-1})\left(\lambda\varphi(t)+(1-\lambda)\varphi(s)\right)\varphi(t_{i+1})\dots\varphi(t_n)$$

$$= \lambda\left(\varphi(t_1)\varphi(t_2)\dots\varphi(t_{i-1})\,\varphi(t)\varphi(t_{i+1})\dots\varphi(t_n)\right)$$

$$+(1-\lambda)\left(\varphi(t_1)\varphi(t_2)\dots\varphi(t_{i-1})\,\varphi(s)\varphi(t_{i+1})\dots\varphi(t_n)\right)$$

$$= \lambda X_{t_n}^i(\varphi(t),\cdot)+(1-\lambda)X_{t_n}^i(\varphi(s),\cdot).$$

But for $\boldsymbol{t} = (1,1,\dots,0), \boldsymbol{s} = (0,1,\dots,1) \in [0,1]^n$, $\varphi(\boldsymbol{t}) = \boldsymbol{t}$, we have

$$X\left(\left(\lambda\varphi(\boldsymbol{t})+(1-\lambda)\varphi(\boldsymbol{s})\right),\cdot\right) = X\left((\lambda\boldsymbol{t}+(1-\lambda)\boldsymbol{s}),\cdot\right)$$

$$= X\left(\left(\lambda,1,1,\dots,(1-\lambda)\right),\cdot\right) = \lambda(1-\lambda)$$

and

$$\lambda X(\varphi(\boldsymbol{t}),\cdot)+(1-\lambda)X(\varphi(\boldsymbol{s}),\cdot) = \lambda X(\boldsymbol{t},\cdot)+(1-\lambda)X(\boldsymbol{s},\cdot) = \lambda.0+(1-\lambda).0 = 0.$$

This gives

$$X\left(\left(\lambda\varphi(\boldsymbol{t})+(1-\lambda)\varphi(\boldsymbol{s})\right),\cdot\right) > \lambda X(\varphi(\boldsymbol{t}),\cdot)+(1-\lambda)X(\varphi(\boldsymbol{s}),\cdot)$$

for all $\lambda \in [0,1]$, that is, X is not φ-convex on $[0,1]^n$.

Remark 3.1. *If $X:\Delta^n \times \Omega \to \mathbb{R}$ is general convex stochastic process on n-coordinates, then $X^i_{t_n}:[\varphi(u_i),\varphi(v_i)]\times\Omega\to\mathbb{R}$ is general convex on $[u_i,v_i]$ and $\varphi:[u_i,v_i]\to[u_i,v_i]$ is mean-square continuous function, $\varphi(u_i)<\varphi(v_i)$, for each $i=1,2,\dots,n$. From Hermite-Hadamard inequality, we have almost everywhere*

$$X^i_{t_n}\left(\frac{\varphi(u_i)+\varphi(v_i)}{2},\cdot\right)\le\frac{1}{\varphi(v_i)-\varphi(u_i)}\int_{\varphi(u_i)}^{\varphi(v_i)}X^i_{t_n}(\varphi(t_i),\cdot)d\varphi(t_i)$$

$$\le\frac{X^i_{t_n}(\varphi(u_i),\cdot)+X^{k+1}_{t_n}(\varphi(v_i),\cdot)}{2},\qquad(1)$$

Following results comprise Hermite-Hadamard's inequality for general convex stochastic processes of first sense on n-coordinates:

Theorem 3.1. *Let $X:\Delta^n\times\Omega\to\mathbb{R}$ be general convex stochastic process on n-coordinates, and $\varphi:[u_i,v_i]\to[u_i,v_i]$ be mean-square continuous function, $\varphi(u_i)<\varphi(v_i)$, for each $i=1,2,\dots,n$. Then we have almost everywhere*

$$\sum_{k=1}^{n}\frac{1}{\varphi(v_k)-\varphi(u_k)}\int_{\varphi(u_k)}^{\varphi(v_k)}X^{k+1}_{t_n}\left(\frac{\varphi(u_{k+1})+\varphi(v_{k+1})}{2},\cdot\right)d\varphi(t_k)$$

$$\le\sum_{k=1}^{n}\frac{1}{(\varphi(v_k)-\varphi(u_k))(\varphi(v_{k+1})-\varphi(u_{k+1}))}\int_{\varphi(u_k)}^{\varphi(v_k)}\int_{\varphi(u_{k+1})}^{\varphi(v_{k+1})}X^{k+1}_{t_n}(\varphi(t_{k+1}),\cdot)\,d\varphi(t_{k+1})\,d\varphi(t_k)$$

$$\le\sum_{k=1}^{n}\frac{1}{2(\varphi(v_k)-\varphi(u_k))}\int_{\varphi(u_k)}^{\varphi(v_k)}\left[X^{k+1}_{t_n}(\varphi(u_{k+1}),\cdot)+X^{k+1}_{t_n}(\varphi(v_{k+1}),\cdot)\right]dt_k.\qquad(2)$$

Proof. By applying Hermite-Hadamard's inequality for general convex stochastic process $X^{i+1}_{t_n}$ on interval $[\varphi(u_{i+1}),\varphi(v_{i+1})]$ we have almost everywhere

$$X^{i+1}_{t_n}\left(\frac{\varphi(u_{i+1})+\varphi(v_{i+1})}{2},\cdot\right)\le\frac{1}{\varphi(v_{i+1})-\varphi(u_{i+1})}\int_{\varphi(u_{i+1})}^{\varphi(v_{i+1})}X^{i+1}_{t_n}(\varphi(t_{i+1}),\cdot)d\varphi(t_{i+1})$$

$$\le\frac{X^{i+1}_{t_n}(\varphi(u_{i+1}),\cdot)+X^{i+1}_{t_n}(\varphi(v_{i+1}),\cdot)}{2}.$$

By integrating

$$\frac{1}{\varphi(v_i)-\varphi(u_i)}\int_{\varphi(u_i)}^{\varphi(v_i)}X^{i+1}_{t_n}\left(\frac{\varphi(u_{i+1})+\varphi(v_{i+1})}{2},\cdot\right)d\varphi(t_i)$$

$$\frac{1}{(\varphi(v_i)-\varphi(u_i))(\varphi(v_{i+1})-\varphi(u_{i+1}))}\int_{\varphi(u_i)}^{\varphi(v_i)}\int_{\varphi(u_{i+1})}^{\varphi(v_{i+1})}X^{i+1}_{t_n}(t_{i+1},\cdot)\,d\varphi(t_{i+1})d\varphi(t_i)$$

$$\le\le\frac{1}{2(\varphi(v_i)-\varphi(u_i))}\int_{\varphi(u_i)}^{\varphi(v_i)}\left(X^{i+1}_{t_n}(\varphi(u_{i+1}),\cdot)+X^{i+1}_{t_n}(\varphi(v_{i+1}),\cdot)\right)d\varphi(t_i).$$

Taking summation from 1 to n we get (2).

Theorem 3.2. *Let $X\colon \Delta^n \times \Omega \to \mathbb{R}$ be general convex stochastic process on n-coordinates, and $\varphi\colon [u_i, v_i] \to [u_i, v_i]$ be mean-square continuous function, $\varphi(u_i) < \varphi(v_i)$, for each $i = 1,2,\dots,n$. Then for $\boldsymbol{u}, \boldsymbol{v} \in \Delta^n$ we have almost everywhere*

$$\sum_{k=1}^{n} \frac{1}{2(\varphi(v_k) - \varphi(u_k))} \int_{\varphi(u_k)}^{\varphi(v_k)} \left(X_{u_n}^k (\varphi(t_k),\cdot) + X_{v_n}^k (\varphi(t_k),\cdot) \right) d\varphi(t_k)$$

$$\leq \frac{n}{2} [X(\varphi(\boldsymbol{u}),\cdot) + X(\varphi(\boldsymbol{v}),\cdot)] + \frac{1}{2} \sum_{k=1}^{n} \left[X_{u_n}^k (\varphi(v_k),\cdot) + X_{v_n}^k (\varphi(u_k),\cdot) \right].$$

Proof. Since $X\colon \Delta^n \times \Omega \to \mathbb{R}$ is general convex stochastic on n-coordinates, there for $X_{t_n}^i : [\varphi(u_i), \varphi(v_i)] \times \Omega \to \mathbb{R}$ is general convex stochastic process on $[\varphi(u_i), \varphi(v_i)]$ for each $i = 1,2,\dots,n$, we have almost everywhere

$$\frac{1}{\varphi(v_i) - \varphi(u_i)} \int_{\varphi(u_i)}^{\varphi(v_i)} X_{u_n}^i (\varphi(t_i),\cdot) d\varphi(t_i) \leq \frac{X(\varphi(\boldsymbol{u}),\cdot) + X_{u_n}^i (\varphi(v_i),\cdot)}{2}$$

$$\frac{1}{\varphi(v_i) - \varphi(u_i)} \int_{\varphi(u_i)}^{\varphi(v_i)} X_{v_n}^i (\varphi(t_i),\cdot) d\varphi(t_i) \leq \frac{X_{v_n}^i (\varphi(u_i),\cdot) + X(\varphi(\boldsymbol{v}),\cdot)}{2}$$

Adding above two inequalities we have

$$\frac{1}{\varphi(v_i) - \varphi(u_i)} \int_{\varphi(u_i)}^{\varphi(v_i)} \left[X_{u_n}^i (\varphi(t_i),\cdot) + X_{v_n}^i (\varphi(t_i),\cdot) \right] d\varphi(t_i)$$

$$\leq \frac{X(\varphi(\boldsymbol{u}),\cdot) + X(\varphi(\boldsymbol{v},\cdot) + X_{u_n}^i (\varphi(v_i),\cdot) + X_{v_n}^i (\varphi(u_i),\cdot)}{2},$$

for $i = 1,2,\dots,n$. This completes the proof.

Corollary 3.1. *Let $X\colon \Delta^2 \times \Omega \to \mathbb{R}$ be general convex stochastic process on 2-coordinates. Then we have almost everywhere*

$$X\left(\left(\frac{\varphi(u_1) + \varphi(v_1)}{2}, \frac{\varphi(u_2) + \varphi(v_2)}{2} \right), \cdot \right)$$

$$\leq \frac{1}{2(\varphi(v_2) - \varphi(u_2))} \int_{\varphi(u_1)}^{\varphi(v_1)} X\left(\left(\varphi(t_1), \frac{\varphi(u_2) + \varphi(v_2)}{2} \right), \cdot \right) d\varphi(t_1)$$

$$+ \frac{1}{2(\varphi(v_2) - \varphi(u_2))} \int_{\varphi(u_2)}^{\varphi(v_2)} X\left(\left(\frac{\varphi(u_1) + \varphi(v_1)}{2}, \varphi(t_2) \right), \cdot \right) d\varphi(t_2)$$

$$\leq \frac{1}{(\varphi(v_1)-\varphi(u_1))(\varphi(v_2)-\varphi(u_2))}\int_{\varphi(u_1)}^{\varphi(v_1)}\int_{\varphi(u_2)}^{\varphi(v_2)} X\big((\varphi(t_1),t_2),\cdot\big)\,d\varphi(t_2)d\varphi(t_1)$$

$$\leq \frac{1}{4(\varphi(v_1)-\varphi(u_1))}\int_{\varphi(u_1)}^{\varphi(v_1)}\Big[X\big((\varphi(t_1),\varphi(u_2)),\cdot\big)+X\big((\varphi(t_1),\varphi(v_2)),\cdot\big)\Big]d\varphi(t_1)$$

$$+\frac{1}{4(\varphi(v_2)-\varphi(u_2))}\int_{\varphi(u_2)}^{\varphi(v_2)}\Big[X\Big(\big(\varphi(u_1),\varphi(t_2)\big),\cdot\Big)+X\Big(\big(\varphi(v_1),\varphi(t_2)\big),\cdot\Big)\Big]\,d\varphi(t_2)$$

$$\leq \frac{X(\varphi(u_1),\varphi(u_2)),\cdot+X\big((\varphi(u_1),\varphi(v_2)),\cdot\big)+X\big((\varphi(v_1),\varphi(u_2)),\cdot\big)+X\big((\varphi(v_1),\varphi(v_2)),\cdot\big)}{4}.$$

Theorem 3.3.*If $X\colon \Delta^n\times\Omega\to\mathbb{R}$ is general convex stochastic process on n-coordinates, then $X_{t_n}^k:[\varphi(u_k),\varphi(v_k)]\times\Omega\to\mathbb{R}$ is general convex on $[\varphi(u_k),\varphi(v_k)]$ for each $k=1,2,\dots,n$. From Hermite-Hadamard inequality, we have almost everywhere*

$$\sum_{k=1}^{n-1} X\left(\left(\varphi(t_1),\dots,\varphi(t_{k-1}),\frac{\varphi(u_k)+\varphi(v_k)}{2},\frac{\varphi(u_{k+1})+\varphi(v_{k+1})}{2},\dots,\varphi(t_n)\right),\cdot\right)$$

$$\leq \sum_{k=1}^{n-1}\frac{1}{\varphi(v_k)-\varphi(u_k)}\int_{\varphi(u_k)}^{\varphi(v_k)} X_{t_n}^k\left(\frac{\varphi(u_{k+1})+\varphi(v_{k+1})}{2},\cdot\right)d\varphi(t_k)$$

$$\leq \sum_{k=1}^{n-1}\frac{1}{(\varphi(v_k)-\varphi(u_k))(\varphi(v_{k+1})-\varphi(u_{k+1}))}\int_{\varphi(u_k)}^{\varphi(v_k)}\int_{\varphi(u_{k+1})}^{\varphi(v_{k+1})} X_{t_n}^{k+1}(\varphi(t_{k+1}),\cdot)d\varphi(t_{k+1})d\varphi(t_k)$$

$$\leq \sum_{k=1}^{n-1}\frac{1}{\varphi(v_k)-\varphi(u_k)}\int_{\varphi(u_k)}^{\varphi(v_k)}\frac{X_{t_n}^k(u_{k+1},\cdot)+X_{t_n}^k(v_{k+1},\cdot)}{2t_k^2}d\varphi(t_k)$$

$$\leq \frac{1}{4}\sum_{k=1}^{n-1}\begin{bmatrix} X\big((\varphi(t_1),\dots,\varphi(t_{k-1}),\varphi(u_k),\varphi(u_{k+1}),\dots,\varphi(t_n)),\cdot\big)\\ +X\big((\varphi(t_1),\dots,\varphi(t_{k-1}),\varphi(v_k),\varphi(u_{k+1}),\dots,\varphi(t_n)),\cdot\big)\\ +X\big((\varphi(t_1),\dots,\varphi(t_{k-1}),\varphi(u_k),\varphi(v_{k+1}),\dots,\varphi(t_n)),\cdot\big)\\ +X\big((\varphi(t_1),\dots,\varphi(t_{k-1}),\varphi(v_k),\varphi(v_{k+1}),\dots,\varphi(t_n)),\cdot\big)\end{bmatrix}. \tag{3}$$

Proof. Since $X\colon\Delta^n\times\Omega\to\mathbb{R}$ is general convex stochastic on n-coordinates, there for $X_{t_n}^i:[\varphi(u_i),\varphi(v_i)]\times\Omega\to\mathbb{R}$ is general convex stochastic process on $[\varphi(u_i),\varphi(v_i)]$ for each $i=1,2,\dots,n$, we have on $[\varphi(u_{i+1}),\varphi(v_{i+1})]$ almost everywhere

$$X_{t_n}^{i+1}\left(\frac{\varphi(u_{i+1})+\varphi(v_{i+1})}{2},\cdot\right)\leq\frac{1}{\varphi(v_{i+1})-\varphi(u_{i+1})}\int_{\varphi(u_{i+1})}^{\varphi(v_{i+1})}X_{t_n}^{i+1}(\varphi(t_{i+1}),\cdot)d\varphi(t_{i+1})$$

$$\leq\frac{X_{t_n}^{i+1}(\varphi(u_{i+1}),\cdot)+X_{t_n}^{i+1}(\varphi(v_{i+1}),\cdot)}{2}.$$

All of sides of the above inequalities by integrating over $[\varphi(u_i),\varphi(v_i)]$

$$\frac{1}{\varphi(v_i)-\varphi(u_i)}\int_{u_i}^{v_i} X_{t_n}^{i+1}\left(\frac{\varphi(u_{i+1})+\varphi(v_{i+1})}{2},\cdot\right)d\varphi(t_i)$$

$$\leq \frac{1}{\big(\varphi(v_i)-\varphi(u_i)\big)\big(\varphi(v_{i+1})-\varphi(u_{i+1})\big)}\int_{\varphi(u_i)}^{\varphi(v_i)}\int_{\varphi(u_{i+1})}^{\varphi(v_{i+1})} X_{t_n}^{i+1}(\varphi(t_{i+1}),\cdot)d\varphi(t_{i+1})d\varphi(t_i)$$

$$\leq \frac{1}{2\big(\varphi(v_i)-\varphi(u_i)\big)}\int_{\varphi(u_i)}^{\varphi(v_i)}\Big(X_{t_n}^{i+1}(\varphi(u_{i+1}),\cdot)+X_{t_n}^{i+1}(\varphi(v_{i+1}),\cdot)\Big)d\varphi(t_i). \quad (4)$$

Again applying the Hermite-Hdamard inequality

$$X\left(\left(\varphi(t_1),\dots,\frac{\varphi(u_i)+\varphi(v_i)}{2},\frac{\varphi(u_{i+1})+\varphi(v_{i+1})}{2},\dots,\varphi(t_n)\right),\cdot\right)$$

$$\leq \frac{1}{\varphi(v_i)-\varphi(u_i)}\int_{u_i}^{v_i} X_{t_n}^{i+1}\left(\frac{\varphi(u_{i+1})+\varphi(v_{i+1})}{2},\cdot\right)d\varphi(t_i) \quad (5)$$

for each $i\in\{1,2,\dots,n-1\}$ and also

$$\frac{1}{2\big(\varphi(v_i)-\varphi(u_i)\big)}\int_{\varphi(u_i)}^{\varphi(v_i)}\Big(X_{t_n}^{i+1}(\varphi(u_{i+1}),\cdot)+X_{t_n}^{i+1}(\varphi(v_{i+1}),\cdot)\Big)d\varphi(t_i)$$

$$=\frac{1}{2}\left[\begin{array}{l}\dfrac{1}{\varphi(v_i)-\varphi(u_i)}\displaystyle\int_{\varphi(u_i)}^{\varphi(v_i)} X_{t_n}^{i+1}(\varphi(u_{i+1}),\cdot)d\varphi(t_i)\\ +\dfrac{1}{\varphi(v_i)-\varphi(u_i)}\displaystyle\int_{\varphi(u_i)}^{\varphi(v_i)} X_{t_n}^{i+1}(\varphi(v_{i+1}),\cdot)d\varphi(t_i)\end{array}\right]$$

$$\leq\frac{1}{2}\left[\begin{array}{l}\dfrac{\left(\begin{array}{l}X\Big((\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(u_i),\varphi(u_{i+1}),\dots,\varphi(t_n)),\cdot\Big)\\ +X\big((\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(v_i),\varphi(u_{i+1}),\dots,\varphi(t_n)),\cdot\big)\end{array}\right)}{2}\\ +\dfrac{\left(\begin{array}{l}X\big((\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(u_i),\varphi(v_{i+1}),\dots,\varphi(t_n)),\cdot\big)\\ +X\big((\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(v_i),\varphi(v_{i+1}),\dots,\varphi(t_n)),\cdot\big)\end{array}\right)}{2}\end{array}\right]$$

$$=\frac{1}{4}\left[\begin{array}{l}X\big((\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(u_i),\varphi(u_{i+1}),\dots,\varphi(t_n)),\cdot\big)\\ +X\big((\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(v_i),\varphi(u_{i+1}),\dots,\varphi(t_n)),\cdot\big)\\ +X\big((\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(u_i),\varphi(v_{i+1}),\dots,\varphi(t_n)),\cdot\big)\\ +X\big((\varphi(t_1),\dots,\varphi(t_{i-1}),\varphi(v_i),\varphi(v_{i+1}),\dots,\varphi(t_n)),\cdot\big)\end{array}\right] \quad (6)$$

for each $i\in\{1,2,\dots,n-1\}$. Using the inequalities (5) and (6) in (4) and taking summation from 1 to $n-1$, we have (3).

Theorem 3.4. *Let X: $\Delta^n \times \Omega \to \mathbb{R}$ be a general convex stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere*

$$X\left(\left(\frac{\varphi(u_1)+\varphi(v_1)}{2},\dots,\frac{\varphi(u_{n-1})+\varphi(v_{n-1}),}{2},\frac{\varphi(u_n)+\varphi(v_n)}{2}\right),\cdot\right)$$

$$\leq\left(\prod_{i=1}^{n}\frac{1}{\varphi(v_i)-\varphi(u_i)}\right)\int_{\varphi(u_1)}^{\varphi(v_1)}\dots\int_{\varphi(u_n)}^{\varphi(v_n)}X_{t_n}^n\left(\varphi(t_n),\cdot\right)d\varphi(t_n)\dots d\varphi(t_1)$$

$$\leq\frac{1}{2^n}\sum_{\delta\in l_i(n)}X(\delta\varphi(u)+(1-\delta)\varphi(v),\cdot), \tag{7}$$

where $l_i(n):=\{\delta\in\mathbb{N}_0^n:\delta\leq 1,|\delta|=n+1-i,i=1,\dots,n+1\}$, $|\delta|:=\delta_1+\cdots+\delta_n\in\mathbb{N}$; $\delta\varphi(u):=\left(\delta_1\varphi(u_1),\dots,\delta_n\varphi(u_n)\right)\in\mathbb{N}_0^n$ for $u,v\in\Delta^n$.

Proof. Using (1), we get the following inequality for $X_{t_n}^n$ almost everywhere

$$X_{t_n}^n\left(\frac{\varphi(u_n)+\varphi(v_n)}{2},\cdot\right)\leq\frac{1}{\varphi(v_n)-\varphi(u_n)}\int_{\varphi(u_n)}^{\varphi(v_n)}X_{t_n}^n\left(\varphi(t_n),\cdot\right)d\varphi(t_n)$$

$$\leq\frac{X_{t_n}^n\left(\varphi(u_n),\cdot\right)+X_{t_n}^n\left(\varphi(v_n),\cdot\right)}{2}.$$

By integrating on $[\varphi(u_{n-1}),\varphi(v_{n-1})]$, we get

$$\frac{1}{\varphi(v_{n-1})-\varphi(u_{n-1})}\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})}X_{t_n}^n\left(\frac{\varphi(u_n)+\varphi(v_n)}{2},\cdot\right)d\varphi(t_{n-1})$$

$$\leq\frac{1}{(\varphi(v_{n-1})-\varphi(u_{n-1}))(\varphi(v_n)-\varphi(u_n))}\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})}\int_{\varphi(u_n)}^{\varphi(v_n)}X_{t_n}^n\left(\varphi(t_n),\cdot\right)d\varphi(t_n)d\varphi(t_{n-1})$$

$$\leq\frac{1}{\varphi(v_{n-1})-\varphi(u_{n-1})}\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})}\frac{X_{t_n}^n\left(\varphi(u_n),\cdot\right)+X_{t_n}^n\left(\varphi(v_n),\cdot\right)}{2}d\varphi(t_{n-1}). \tag{8}$$

From (4), (5), respectively

$$X\left(\left(t_1,\dots,\frac{\varphi(u_{n-1})+\varphi(v_{n-1})}{2},\frac{\varphi(u_n)+\varphi(v_n)}{2}\right),\cdot\right)$$

$$\leq\frac{1}{\varphi(v_{n-1})-\varphi(u_{n-1})}\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})}X_{t_n}^n\left(\frac{\varphi(u_n)+\varphi(v_n)}{2},\cdot\right)d\varphi(t_{n-1}), \tag{9}$$

$$\frac{1}{\varphi(v_{n-1})-\varphi(u_{n-1})}\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})}\left(\frac{X_{t_n}^n\left(\varphi(u_n),\cdot\right)+X_{t_n}^n\left(\varphi(v_n),\cdot\right)}{2}\right)d\varphi(t_{n-1})$$

$$= \frac{1}{2}\left[\begin{array}{l} \dfrac{1}{\varphi(v_{n-1})-\varphi(u_{n-1})}\displaystyle\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})} X_{t_n}^{n}\left(\varphi(u_n),\cdot\right)d\varphi(t_{n-1}) \\ +\dfrac{1}{\varphi(v_{n-1})-\varphi(u_{n-1})}\displaystyle\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})} X_{t_n}^{n}\left(\varphi(v_n),\cdot\right)d\varphi(t_{n-1}) \end{array}\right]$$

$$\leq \frac{1}{2^2}\left[\begin{array}{l} X\big((\varphi(t_1),\dots,\varphi(u_{n-1}),\varphi(u_n)),\cdot\big)+X\big((\varphi(t_1),\dots,\varphi(v_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(u_{n-1}),\varphi(v_n)),\cdot\big)+X\big((\varphi(t_1),\dots,\varphi(v_{n-1}),\varphi(v_n)),\cdot\big) \end{array}\right]. \quad (10)$$

From (8)-(10)

$$X\left(\left(\varphi(t_1),\dots,\frac{\varphi(u_{n-1})+\varphi(v_{n-1})}{2},\frac{\varphi(u_n)+\varphi(v_n)}{2}\right),\cdot\right)$$

$$\leq \frac{1}{(\varphi(v_{n-1})-\varphi(u_{n-1}))(\varphi(v_n)-\varphi(u_n))}\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})}\int_{\varphi(u_n)}^{\varphi(v_n)} X_{t_n}^{n}\left(\varphi(t_n),\cdot\right)d\varphi(t_n)d\varphi(t_{n-1})$$

$$\leq \frac{1}{2^2}\left[\begin{array}{l} X\big((\varphi(t_1),\dots,\varphi(u_{n-1}),\varphi(u_n)),\cdot\big)+X\big((\varphi(t_1),\dots,\varphi(v_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(u_{n-1}),\varphi(v_n)),\cdot\big)+X\big((\varphi(t_1),\dots,\varphi(v_{n-1}),\varphi(v_n)),\cdot\big) \end{array}\right]. \quad (11)$$

Integrating on $[\varphi(u_{n-2}),\varphi(v_{n-2})\,]$

$$\frac{1}{\varphi(v_{n-2})-\varphi(u_{n-2})}\int_{\varphi(u_{n-2})}^{\varphi(v_{n-2})} X\left(\left(\varphi(t_1),\dots,\frac{\varphi(u_{n-1})+\varphi(v_{n-1})}{2},\frac{\varphi(u_n)+\varphi(v_n)}{2}\right),\cdot\right)d\varphi(t_{n-2})$$

$$\leq \left(\prod_{i=n-2}^{n}\frac{1}{\varphi(v_i)-\varphi(u_i)}\right)\int_{\varphi(u_{n-2})}^{\varphi(v_{n-2})}\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})}\int_{\varphi(u_n)}^{\varphi(v_n)} X_{t_n}^{n}\left(\varphi(t_n),\cdot\right)d\varphi(t_n)d\varphi(t_{n-1})d\varphi(t_{n-2})$$

$$\leq \frac{1}{\varphi(v_{n-2})-\varphi(u_{n-2})}\int_{\varphi(u_{n-2})}^{\varphi(v_{n-2})}\frac{1}{2^2}\left[\begin{array}{l} X\big((\varphi(t_1),\dots,\varphi(u_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\left(\left(\varphi(t_1),\dots,\varphi(u_{n-1}),\varphi(v_n)\right),\cdot\right) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-1}),\varphi(v_n)),\cdot\big) \end{array}\right]d\varphi(t_{n-2}). \quad (12)$$

From (4), (5), respectively

$$X\left(\left(\varphi(t_1),\dots,\frac{\varphi(u_{n-2})+\varphi(v_{n-2})}{2},\frac{\varphi(u_{n-1})+\varphi(v_{n-1})}{2},\frac{\varphi(u_n)+\varphi(v_n)}{2}\right),\cdot\right)$$

$$\leq \frac{1}{\varphi(v_{n-2})-\varphi(u_{n-2})}\int_{\varphi(u_{n-2})}^{\varphi(v_{n-2})} X\left(\left(\varphi(t_1),\dots,\frac{\varphi(u_{n-1})+\varphi(v_{n-1})}{2},\frac{\varphi(u_n)+\varphi(v_n)}{2}\right),\cdot\right)d\varphi(t_{n-2}) \quad , (13)$$

$$\leq \frac{1}{\varphi(v_{n-2})-\varphi(u_{n-2})}\int_{\varphi(u_{n-2})}^{\varphi(v_{n-2})}\frac{1}{2^2}\begin{bmatrix} X\big((\varphi(t_1),\dots,\varphi(u_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\Big(\big(\varphi(t_1),\dots,\varphi(u_{n-1}),\varphi(v_n)\big),\cdot\Big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-1}),\varphi(v_n)),\cdot\big)\end{bmatrix}d\varphi(t_{n-2})$$

$$\leq \frac{1}{2^3}\begin{bmatrix} X\big((\varphi(t_1),\dots,\varphi(u_{n-2}),\varphi(u_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-2}),\varphi(u_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(u_{n-2}),\varphi(v_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-2}),\varphi(v_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(u_{n-2}),\varphi(u_{n-1}),\varphi(v_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-2}),\varphi(u_{n-1}),\varphi(v_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(u_{n-2}),\varphi(v_{n-1}),\varphi(v_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-2}),\varphi(v_{n-1}),\varphi(v_n)),\cdot\big)\end{bmatrix}. \qquad (14)$$

From (12)-(14)

$$\frac{1}{\varphi(v_{n-2})-\varphi(u_{n-2})}\int_{\varphi(u_{n-2})}^{\varphi(v_{n-2})} X\left(\left(\varphi(t_1),\dots,\frac{\varphi(u_{n-1})+\varphi(v_{n-1})}{2},\frac{\varphi(u_n)+\varphi(v_n)}{2}\right),\cdot\right)d\varphi(t_{n-2})$$

$$\leq\left(\prod_{i=n-2}^{n}\frac{1}{\varphi(v_i)-\varphi(u_i)}\right)\int_{\varphi(u_{n-2})}^{\varphi(v_{n-2})}\int_{\varphi(u_{n-1})}^{\varphi(v_{n-1})}\int_{\varphi(u_n)}^{\varphi(v_n)} X_{t_n}^{n}(\varphi(t_n),\cdot)d\varphi(t_n)d\varphi(t_{n-1})d\varphi(t_{n-2})$$

$$\leq \frac{1}{2^3}\begin{bmatrix} X\big((\varphi(t_1),\dots,\varphi(u_{n-2}),\varphi(u_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-2}),\varphi(u_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(u_{n-2}),\varphi(v_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-2}),\varphi(v_{n-1}),\varphi(u_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(u_{n-2}),\varphi(u_{n-1}),\varphi(v_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-2}),\varphi(u_{n-1}),\varphi(v_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(u_{n-2}),\varphi(v_{n-1}),\varphi(v_n)),\cdot\big) \\ +X\big((\varphi(t_1),\dots,\varphi(v_{n-2}),\varphi(v_{n-1}),\varphi(v_n)),\cdot\big)\end{bmatrix}.$$

Using inductive method for $n=k-1$, we obtain

$$X\left(\left(\frac{\varphi(u_1)+\varphi(v_1)}{2},\dots,\frac{\varphi(u_{k-2})+\varphi(v_{k-2}),}{2},\frac{\varphi(u_{k-1})+\varphi(v_{k-1})}{2}\right),\cdot\right)$$

$$\leq\left(\prod_{i=1}^{k-1}\frac{1}{\varphi(v_i)-\varphi(u_i)}\right)\int_{\varphi(u_1)}^{\varphi(v_1)}\dots\int_{\varphi(u_{k-1})}^{\varphi(v_{k-1})} X_{t_{k-1}}^{k-1}(\varphi(t_{k-1}),\cdot)d\varphi(t_{k-1})\dots d\varphi(t_1)$$

$$\leq \frac{1}{2^{k-1}} \sum_{\delta \in l_i(k-1)} X(\delta\varphi(u) + (1-\delta)\varphi(v),\cdot),$$

where $l_i(k-1) \coloneqq \{\delta \in \mathbb{N}_0^n : \delta \leq 1, |\delta| = k-i;\ i = 1,\dots,k, k \in \mathbb{N}\}$, $|\delta| \coloneqq \delta_1 + \cdots + \delta_n \in \mathbb{N}$; $\delta u := (\delta_1 u_1, \dots, \delta_n u_n) \in \mathbb{N}_0^n$ for $u, v \in \Delta^n$. Consequently, for $n = k$, we get (7).

Example 1. *Let X: $\Delta^3 \times \Omega \to \mathbb{R}$ be a general convex stochastic process and be integrated in mean-square on Δ^3. Then almost everywhere*

$$X\left(\left(\frac{\varphi(u_1)+\varphi(v_1)}{2}, \frac{\varphi(u_2)+\varphi(v_2)}{2}, \frac{\varphi(u_3)+\varphi(v_3)}{2}\right),\cdot\right)$$

$$\leq \left(\prod_{i=n-2}^{3} \frac{1}{\varphi(v_i)-\varphi(u_i)}\right) \int_{\varphi(u_1)}^{\varphi(v_1)} \int_{u_2}^{\varphi(v_2)} \int_{u_3}^{\varphi(v_3)} X\left((\varphi(t_1), \varphi(t_2), \varphi(t_3)),\cdot\right) d\varphi(t_3) d\varphi(t_2) d\varphi(t_1)$$

$$\leq \frac{1}{2^3} \begin{bmatrix} X\big((\varphi(u_1), \varphi(u_2), \varphi(u_3)),\cdot\big) + X\big((\varphi(v_1), \varphi(u_2), \varphi(u_3)),\cdot\big) \\ +X\big((\varphi(u_1), \varphi(v_2), \varphi(u_3)),\cdot\big) + X\big((\varphi(v_1), \varphi(v_2), \varphi(u_3)),\cdot\big) \\ +X\big((\varphi(u_1), \varphi(u_2), \varphi(v_3)),\cdot\big) + X\big((\varphi(v_1), \varphi(u_2), \varphi(v_3)),\cdot\big) \\ +X\big((\varphi(u_1), \varphi(v_2), \varphi(v_3)),\cdot\big) + X\big((\varphi(v_1), \varphi(v_2), \varphi(v_3)),\cdot\big) \end{bmatrix}. \qquad (13)$$

Proof. According to Theorem 3.4 for n = 3, we get

$$X_{t_3}^3(\varphi(t_3),\cdot) \coloneqq X\big((\varphi(t_1), \varphi(t_2), \varphi(t_3)),\cdot\big)$$

and $l_i(3) \coloneqq \{\delta \in \mathbb{N}_0^3 : \delta \leq 1, |\delta| = 4-i\}, i = 1,2,3,4$. Then

$$l_1(3) = \{(1,1,1)\};\ l_2(3) = \{(0,1,1)\ , (1,0,1), (1,1,0)\},$$

$$l_3(3) = \{(0,0,1)\ , (0,1,0), (1,0,0), \}; l_4(3) = \{(0,0,0)\},$$

and for $\varphi(u) = \big(\varphi(u_1), \varphi(u_2), \varphi(u_3)\big)$, $\varphi(v) = \big(\varphi(v_1), \varphi(v_2), \varphi(v_3)\big)$

$$\sum_{\delta \in l_1(3)} X(\delta\varphi(u) + (1-\delta)\varphi(v),\cdot)$$

$$= X\left((1,1,1)\big(\varphi(u_1), \varphi(u_2), \varphi(u_3)\big) + [(1,1,1) - (1,1,1)]\big(\varphi(v_1), \varphi(v_2), \varphi(v_3)\big),\cdot\right)$$

$$= X\left(\big(\varphi(u_1), \varphi(u_2), \varphi(u_3)\big),\cdot\right);$$

$$\sum_{\delta \in l_2(3)} X(\delta\varphi(u) + (1-\delta)\varphi(v),\cdot)$$

$$= X\left((0,1,1)\big(\varphi(u_1), \varphi(u_2), \varphi(u_3)\big) + [(1,1,1) - (0,1,1)]\big(\varphi(v_1), \varphi(v_2), \varphi(v_3)\big),\cdot\right)$$

$$+ X\left((1,0,1)\big(\varphi(u_1), \varphi(u_2), \varphi(u_3)\big) + [(1,1,1) - (1,0,1)]\big(\varphi(v_1), \varphi(v_2), \varphi(v_3)\big),\cdot\right)$$

$$+X\left((1,1,0)\left(\varphi(u_1),\varphi(u_2),\varphi(u_3)\right)+[(1,1,1)-(1,1,0)]\left(\varphi(v_1),\varphi(v_2),\varphi(v_3)\right),\cdot\right)$$

$$=X\left(\left(\varphi(u_1),\varphi(v_2),\varphi(v_3)\right),\cdot\right)+X\left(\left(\varphi(u_1),\varphi(v_2),\varphi(u_3)\right),\cdot\right)$$

$$+X\left(\left(\varphi(u_1),\varphi(u_2),\varphi(v_3)\right),\cdot\right);$$

So,

$$\sum_{\delta\in l_3(3)}X(\delta\varphi(u)+(1-\delta)\varphi(v),\cdot)=X\left(\left(\varphi(v_1),\varphi(v_2),\varphi(u_3)\right),\cdot\right)$$

$$+X\left(\left(\varphi(v_1),\varphi(u_2),\varphi(v_3)\right),\cdot\right)+X\left(\left(\varphi(u_1),\varphi(v_2),\varphi(v_3)\right),\cdot\right);$$

$$\sum_{\delta\in l_4(3)}X(\delta\varphi(u)+(1-\delta)\varphi(v),\cdot)=X\left(\left(\varphi(v_1),\varphi(v_2),\varphi(v_3)\right),\cdot\right).$$

Thus

$$\sum_{\delta\in l_i(3)}X(\delta\varphi(u)+(1-\delta)\varphi(v),\cdot)$$

$$=X\left(\left(\varphi(u_1),\varphi(u_2),\varphi(u_3)\right),\cdot\right)+X\left(\left(\varphi(u_1),\varphi(v_2),\varphi(v_3)\right),\cdot\right)$$

$$+X\left(\left(\varphi(u_1),\varphi(v_2),\varphi(u_3)\right),\cdot\right)+X\left(\left(\varphi(u_1),\varphi(u_2),\varphi(v_3)\right),\cdot\right)$$

$$+X\left(\left(\varphi(v_1),\varphi(v_2),\varphi(u_3)\right),\cdot\right)+X\left(\left(\varphi(v_1),\varphi(u_2),\varphi(v_3)\right),\cdot\right)$$

$$+X\left(\left(\varphi(u_1),\varphi(v_2),\varphi(v_3)\right),\cdot\right)+X\left(\left(\varphi(v_1),\varphi(v_2),\varphi(v_3)\right),\cdot\right).$$

Using all of the above equalities in (13), we obtain the desired result in this example.

4. CONCLUSION

In this paper, we obtain some new Hermite-Hadamard type inequalities for φ-convex stochastic processes on n-coordinates. As special cases, one can obtain several new and correct versions of the previously known results for various classes of these stochastic processes.

ACKNOWLEDGEMENTS

The author would like to thank to Assoc. Prof. İmdat İŞCAN, Giresun University, Department of Mathematics, owing to his ideas.

References

[1] Alomari, M. W., A generalization of Hermite–Hadamard's inequality, Kragujevac J. Math., 41(2) (2017) 313–328.

[2] Mwaeze, E. R., Generalized Hermite–Hadamard's inequality for functions convex on the coordinates, Applied Mathematics E-Notes, 18 (2018) 275-283.

[3] Nikodem, K., On convex stochastic processes, Aequat. Math., 20 (1980) 184-197.

[4] Shaked, M., Shanthikumar, J.G., Stochastic convexity and its applications, Advances in Applied Probability, 20 (1988) 427-446.

[5] Skowronski, A., On some properties of J-convex stochastic processes, Aequat. Math., 44 (1992) 249-258.

[6] Kotrys, D., Hermite-Hadamard inequality for convex stochastic processes, Aequat. Math., 83, (2012) 143-151.

[7] Set, E., Sarıkaya, M.Z., Tomar, M., Hermite-Hadamard inequalities for coordinates convex stochastic processes, Mathematica Aeterna, 5 (2) (2015) 363-382.

[8] Okur, N. İşcan, İ. Usta, Y., Some Integral inequalities for harmonically convex stochastic processes on the coordinates, Advanced Math. Models & Applications, 3(1) (2018) 63-75.

[9] Sarikaya, M.Z., Kiriş, M.E., Çelik, N., Hermite-Hadamard type inequality for φ_h-convex stochastic processes, AIP Conference Proceedings 1726, 020076, 2016; doi: 10.1063/1.4945902

[10] Cristescu, G., Hadamard type inequalities for φ-convex functions, Annals of the University of Oradea, Fascicle of Management and Technological Engineering, CD-Rom Edition, 12 (2004), no. 3.

[11] Sarikaya, M.Z., On Hermite Hadamard-type inequalities for φ_h-convex functions, Kochi J. of Math., 9 (2014) 83-90.

[12] Sarikaya, M.Z., On Hermite Hadamard-type inequalities for strongly φ-convex functions, Southeast Asian Bull. Math., 39 (1) (2015) 123-132.

[13] Sarikaya, M.Z., Set, E., Ozdemir, M.E., Dragomir, S.S., New some Hadamard's type inequalities for co-ordinated convex functions, Tamsui Oxford Journal of Information and Mathematical Sciences, 28 (2) (2012) 137-152.

[14] Karahan, V., Okur, N., Hermite-Hadamard Type Inequalities for Convex Stochastic Processes On n-Coordinates, Special Issue: Proceedings of International Conference on Mathematicd and Mathematics Education (ICMME 2018), Turk. J. Math. Comput. Sci., 10 (2018) 256-262.